DUALITY IN FRACTIONAL PROGRAMMING*

Finn Kydland

Graduate School of Industrial Administration
Carnegie-Mellon University
Pittsburgh, Pennsylvania

ABSTRACT

This paper develops theoretical and computational aspects of the dual problem in linear fractional programming. This is done on the basis of two alternative methods for solving the primal fractional programming problem, both of which were presented in earlier literature. Parametric changes in the resource-vector are considered, and attention is given to infinitesimal as well as to discrete changes.

INTRODUCTION

Consider the fractional programming problem

$$\text{maximize } Q(x)=\frac{N(x)}{D(x)}=\frac{cx+\alpha}{dx+\beta}$$

subject to

(P1)
$$\begin{aligned} Ax&=b \\ x&\geqslant 0. \end{aligned} \qquad x\epsilon R^n$$

The first methods for solving this problem were apparently proposed by Dinkelbach [3] and Charnes and Cooper [2], of which the latter is the easiest to use. It applies the simplex-method to a simple transformation of the original problem (P1), and the problem can therefore be solved using a regular simplex-algorithm for linear programs.

Martos [10] suggested an algorithm for applying the simplex-method to the original problem (P1), but with modified entry criterion based on the gradient.† Swarup [12] describes an algorithm equivalent to Martos' and also suggests how to solve (P1) when the variables are required to be integers.

Some results connecting the solutions of linear fractional programming and those of parametric linear programming are obtained in Jagannathan [5] and Dinkelbach [4]. In Kydland [8] we describe an algorithm based on a parametric programming formulation of (P1). The article also contains an application of fractional programming to linear operations.

Some work has been done on formulating a dual problem to (P1) and calculating dual variables. Swarup considered theoretical aspects in [13], and went on to compute dual variables in [14] and [15].

*This report was prepared as part of the activities of the Management Sciences Research Group at Carnegie-Mellon University's Graduate School of Industrial Administration. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

†Wagner and Yuan [16] have shown that this algorithm is algorithmically equivalent to Charnes and Cooper's method, in the sense that the variables of (P1) enter the basis in the same order.

However, he has a pair of dual variables for each constraint, one being dual with respect to the numerator and one for the denominator of the objective function. Chadda [1] in his decomposition algorithm implicitly computes the same pair of dual variables as Swarup does.

Kaska [6] found the correct formulation of the dual variables; however, he incorrectly gives the dual variables the interpretation of evaluating unit changes of the resource-vector b instead of infinitesimal changes. The reason for this difference from linear programming is, of course, that the objective function is nonlinear.

In the next section we will state assumptions and formulate a dual problem to (P1). Then a section is devoted to a brief description of Charnes and Cooper's solution method, and we explain how to compute the dual variables based on their method. A comparison with Martos' algorithm is offered, and here attention is also given to discrete changes of the resource vector b. The last section contains a numerical example.

PRELIMINARIES

Referring to (P1), let $S = \{x | Ax = b,\ x \geqslant 0\}$. We make the following assumptions

(a)
$$S \neq \emptyset$$

(b)
$$S \cap \{x | D(x) = 0\} = \emptyset$$

Without restricting the problem, we may also assume that $D(x) > 0$. If our original function is such that $D(x) < 0$, then we can instead maximize $Q(x) = \dfrac{-N(x)}{-D(x)}$, where $-D(x) > 0$.

According to Wolfe [17] the dual problem to (P1) is

(D1)
$$\text{minimize } Q(x) - u(Ax - b) + vIx$$

subject to:
$$\nabla Q(x) - uA + vI = 0$$

(1)
$$v \geqslant 0.$$

In the following we will make use of the Kuhn-Tucker necessary optimality theorem for differentiable functions.* Since our constraints are all linear, (P1) clearly satisfies the constraint qualification. We therefore have:

THEOREM 1: Let $\bar{x}$ solve (P1). Then there exists $\bar{u} \in R^m$ and $\bar{v} \in R^n$ such that

(KT)
$$\nabla Q(\bar{x}) - \bar{u}A + \bar{v}I = 0$$
$$\bar{u}(A\bar{x} - b) - \bar{v}I\bar{x} = 0$$
$$\bar{v} \geqslant 0.$$

In view of Theorem 1 we see that at the optimum the value of the objective function of (D1) becomes

*See Kuhn and Tucker [7] or Mangasarian [9].

$$Q(\bar{x}) - \bar{u}(A\bar{x} - b) + \bar{v}I\bar{x} = Q(\bar{x});$$

i.e., the same as for (P1).

It may also be noted that since $A\bar{x} = b$, (KT) implies

$$x_j > 0 \Rightarrow \frac{\partial Q(\bar{x})}{\partial x_j} - \bar{u}a_j = 0.$$

In the following, we will show how the dual variables can be found using first the method proposed by Charnes and Cooper [2], and then the algorithm by Martos [10].

CHARNES AND COOPER'S METHOD

We make the transformation

$$y \equiv tx,$$

where $t = \dfrac{1}{dx + \beta}$. Then the primal problem (P1) can be written

$$\text{(P2)} \quad \begin{array}{ll} \text{maximize} & L(y, t) \equiv cy + \alpha t \\ \text{subject to} & \end{array}$$

$$Ay - bt = 0 \tag{2}$$

$$dy + \beta t = 1 \tag{3}$$

$$y \geqslant 0, \qquad t \geqslant 0.$$

The dual of (P2) is

$$\text{(D2)} \quad \begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \end{array}$$

$$wA + \lambda d \geqslant c \tag{4}$$

$$-wb + \lambda B \geqslant \alpha.$$

From linear programming theory we know that if $(\bar{y}, \bar{t})$ is an optimal solution to (P2), then there exists $(\bar{w}, \bar{\lambda})$ optimizing (D2) and such that $\bar{\lambda} = L(\bar{y}, \bar{t}) = Q(\bar{x})$, where $\bar{x} = \bar{y}/\bar{t}$. This means that the dual variable corresponding to constraint (3) has the value of the maximand we are looking for. What is then the relationship between w in (D2) and u in (D1)? Let us write (1) more explicitly

$$\frac{1}{dx + \beta}\left(c - \frac{cx + \alpha}{dx + \beta}\, d\right) - uA + vI = 0,$$

i.e., $uA + \dfrac{cx + \alpha}{dx + \beta}\, td \geqslant tc$, or

$$\frac{1}{t}uA + \frac{cx+\alpha}{dx+\beta}d \geqslant c. \tag{5}$$

At the optimum $\lambda = \dfrac{cx+\alpha}{dx+\beta}$, and so (5) is equivalent to (4) if $w = \dfrac{1}{t}u$, or $u = tw$.

CONCLUSION: Solving the problem (P2) will give us one dual variable corresponding to each constraint. The dual variables of (P1) are found by multiplying the dual variables for the constraints (2) by t. The dual variable for constraint (3) has the value at the optimum of the function $Q(x)$ that we are maximizing.

MARTOS' ALGORITHM

This algorithm is based on a direct application of the simplex-algorithm, with the exception that the entry criterion for changing the basis is different. The algorithm is well explained in Martos [10] and we will only report our results here.

Assume that the algorithm has been applied to the problem (P1). Then the following quantities can be found in the final tableau

$$\bar{c}^B = c^B B^{-1} \qquad \text{and} \qquad \bar{d}^B = d^B B^{-1},$$

where B is the final basis, and c^B and d^B denote the subvectors of c and d corresponding to vectors in the basis. $\bar{c}^B$ and $\bar{d}^B$ are similar to dual variables in linear programming, except that here we have two sets of them, one set corresponding to the numerator and one to the denominator.

By use of Theorem 1 it is easy to show Theorem 2.

THEOREM 2: If (P1) has a nondegenerate optimal solution $\bar{x}$, then

$$\bar{u} = (\bar{c}^B - Q(\bar{x})\, \bar{d}^B)/D(\bar{x})\,. \tag{6}$$

Since the objective functi n is nonlinear, the u_i's will, in general, give the rate of change of the objective function only for infinitely small changes in the b_i's. An advantage with Martos' algorithm in terms of postoptimization is that one can easily calculate the change in Q for changes in b by any given amounts that do not result in a change of basis, or in general an upper bound for ΔQ. The formula when b changes to $b+h$ is

$$\Delta Q = \frac{N + \bar{c}^B h}{D + \bar{d}^B h} - \frac{N}{D} = \frac{(\bar{c}^B - Q\bar{d}^B)h}{D + \bar{d}^B h} = \frac{D\bar{u}h}{D + \bar{d}^B h}. \tag{7}$$

When Charnes and Cooper's method is used, the change in b will show up as a change in the column corresponding to the variable, t, in the coefficient matrix for the LP-program, and we know that this column will always be in the basis. It is therefore clear that analysis of changes in b here becomes much more laborious.*

---

*See Simonnard [11], p. 157.

NUMERICAL EXAMPLE: Consider the example

$$\text{maximize } Q(x) = \frac{N(x)}{D(x)} = \frac{3x_1 + 5x_2 + 2x_3 - 1}{x_1 + 3x_2 + x_3 + 5}$$

subject to:

$$3x_1 + 2x_2 + x_3 + x_4 = 2$$

$$x_1 + 4x_2 + x_3 + x_5 = 1.$$

Following Charnes and Cooper, we let $t \equiv \dfrac{1}{x_1 + 3x_2 + x_3 + 5}$, and formulate the problem as follows

$$\text{maximize } 3y_1 + 5y_2 + 2y_3 - t,$$

subject to:

$$3y_1 + 2y_2 + y_3 - 2t + y_4 = 0$$

$$y_1 + 4y_2 + y_3 - t + y_5 = 0$$

$$y_1 + 3y_2 + y_3 + 5t = 1.$$

Applying the simplex-method, we obtain the following final tableau:

c_j^B \ c_j		3	6	2	-1	0	0	
		y_1	y_2	y_3	t	y_4	y_5	
3	y_1	1	$-\frac{13}{12}$	0	0	$\frac{1}{2}$	$-\frac{7}{12}$	$\frac{1}{12}$
2	y_3	0	$\frac{59}{12}$	1	0	$-\frac{1}{2}$	$\frac{17}{12}$	$\frac{1}{12}$
-1	t	0	$-\frac{1}{6}$	0	1	0	$-\frac{1}{6}$	$\frac{1}{6}$
$z_j - c_j$		0	$\frac{21}{12}$	0	0	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{4}$

This tableau is reached after three iterations starting with basis variables y_4, y_5, and t. The solution to the original problem is

$$x_1 = \frac{\bar{y}_1}{\bar{t}} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$$

$$x_2 = \frac{\bar{y}_2}{\bar{t}} = 0$$

$$x_3 = \frac{\bar{y}_3}{\bar{t}} = \frac{1}{2} \cdot$$

The dual variables above are*

$$\bar{w}_1 = \frac{1}{2}, \ \bar{w}_2 = \frac{5}{4}, \ \bar{\lambda} = \frac{1}{4} \cdot$$

Therefore,

$$\bar{u}_1 = \bar{t}\bar{w}_1 = \frac{1}{12}$$

$$\bar{u}_2 = \bar{t}\bar{w}_2 = \frac{5}{24}$$

$$Q(\bar{x}) = \bar{\lambda} = \frac{1}{4} \cdot$$

Martos' algorithm would, of course, have given the same solution, with dual variables calculated from (6). If, however, we want to know the change in Q if for instance b_2 increases by one unit, we can use formula (7), which gives the result $\Delta Q = \frac{5}{28} \neq \bar{u}_2$.

REFERENCES

[1] Chadda, S. S., "A Decomposition Principle for Fractional Programming," Opsearch, Vol. 4 (1967).
[2] Charnes, A. and W. W. Cooper, "Programming with Linear Fractional Functionals," Nav. Res. Log. Quart. Vol. 9, No. 3 and 4 (1962).
[3] Dinkelbach, W., "Die Maximierung eines Quotienten zweier linearer Funktionen unter Linearen Nebenbedingungen," Zietschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete Vol. 1 (1962).
[4] Dinkelbach, W., "On Nonlinear Fractional Programming," Management Science Vol. 13 (1967).

*Notation as in the section describing the algorithm.

[5] Jagannathan, R., "On some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming," Management Science Vol. 12 (1966).

[6] Kaska, J., "Duality in Linear Fractional Programming," Economicko-Matematicky Obzor Vol. 5, No. 4 (1969).

[7] Kuhn, H. W., and A. W. Tucker, "Nonlinear Programming" (J. Neyman, ed.), Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley (1950).

[8] Kydland, F., "Simulation of Linear Operations," Institute for Shipping Research, Norwegian School of Economics and Business Administration, Bergen; translated and reprinted from *Sosialoekonomen*, Vol. 23 (1969).

[9] Mangasarian, O. S., *Nonlinear Programming* (McGraw-Hill, 1969).

[10] Martos, B., "Hyperbolic Programming," Nav. Res. Log. Quart. Vol. 11 (1964).

[11] Simonnard, M., *Linear Programming* (Prentice Hall, 1966).

[12] Swarup, K., "Some Aspects of Linear Fractional Functionals Programming," Austr. Journal of Stat. Vol. 7 (1965).

[13] Swarup, K., "Some Aspects of Duality in Fractional Programming," Zeitschrift fur Angewandte Mathematik and Mechanik, Vol. 47 (1967).

[14] Swarup, K., "Duality in Fractional Programming," Unternehmensforschung, Vol. 12 (1968).

[15] Swarup, K., "Duality for Transportation Problem in Fractional Programming," Cahiers du Centre d'Etudes de Recherche Operationelle, Vol. 10 (1968).

[16] Wagner, H. M. and J. S. C. Yuan, "Algorithmic Equivalence in Linear Fractional Programming," Management Science Vol. 14 (1968).

[17] Wolfe, P., "A Duality Theorem for Nonlinear Programming," Quarterly of Applied Mathematics Vol. 19 (1961).